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Solution of master equations for small bistable systems

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Abstract. The master equations for bistability in the bad cavity limit in an atomic system is solved when the number of atoms N is not more than 45. A detailed comparison is made of results from Fokker-Planck equations and from the master equation. The validity of the decorrelation approximation for the master equation is examined.

1. Introduction

The theory of absorptive optical bistability for homogeneously broadened two-level atoms interacting resonantly with a coherent field has now had good experimental support at a quantitative level [1]. Most of the predictions have been based on the solutions of the Maxwell-Bloch equations [2] which do not take into account fluctuations in the electromagnetic field. However, for these experiments the saturation photon number is of the order of 10^5, and so quantum statistical effects are either not important or can be adequately treated through a linearised analysis of fluctuations around steady states. This linearised analysis can be done by using a Fokker-Planck equation (FPE) [3, 4] or by making suitable factorisation assumptions on correlation functions [5] (decorrelation method). A linearised analysis of fluctuations (where it exists) shows that the strength of fluctuations scales inversely with the saturation photon number [3] N_s or the number of atoms [4] N. At critical points, such as the threshold for hysteresis, a linearised theory does not exist and then it is necessary to perform a non-linear [6] analysis of quantum fluctuations using the FPE. However, a description in terms of FPE still requires large N or N_s. If the saturation photon numbers in these experiments could be reduced to the order of 10, we might extrapolate the results from theories valid for large N or N_s and state that quantum statistical effects could be of the order of 10%. We wish to examine how far such extrapolations are valid in practice. The physical domains where fluctuations become large, although difficult to achieve, are potentially feasible experimentally.

An approach to non-equilibrium quantum statistical systems which is general and does not depend on N or N_s being large is based on quantum mechanical master equations (ME) which are Schrödinger equations for the density matrix (when damping has been incorporated by considering coupling to reservoir modes). This coupling is assumed to be weak and its spectrum flat. The weak coupling assumption allows a calculation to second order in this coupling. These standard assumptions [7] are

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physically reasonable (even for small systems) and give rise to the well known theory of quantum Markov processes. The ME is difficult to solve directly and until now has been solved (in the bad cavity limit) explicitly only for $N=1$, or for the steady state when there is only collective damping [8, 9]. We will solve the ME in this limit for N of the order of 45 by making use of important relations due to permutation symmetry. (The size of systems that can be treated this way will increase as more powerful computers become available.) Without the use of this symmetry, calculations for $N<5$ only would be feasible.

For a general quality cavity and in a frame rotating at the resonant frequency ω we have (in the absence of detuning) the ME [10, 11]

$$\dot{\rho}=\alpha[a^+-a,\rho]+g[a^+J^+-aJ^-,\rho]+\tfrac{1}{2}\gamma_\parallel\sum_{j=1}^{N}(2\sigma_{(j)}^-\rho\sigma_{(j)}^+-\sigma_{(j)}^+\sigma_{(j)}^-\rho-\rho\sigma_{(j)}^+\sigma_{(j)}^-)$$

$$+2\gamma_0\sum_{j=1}^{N}(\sigma_{(j)}^z\rho\sigma_{(j)}^z-\rho)+\kappa(2a\rho a^+-a^+a\rho-\rho a^+a) \tag{1}$$

where a^+ and a are photon creation and annihilation operators, g is the atom–field coupling constant ($g\equiv(\omega\mu^2/2\hbar\varepsilon_0V)^{1/2}$), V is the quantisation volume, μ is the strength of the atomic dipole moment and ε_0 is the vacuum permittivity. The pseudo-spin operators describing the two-level atoms are the Pauli spin matrices $\sigma_{(j)}^{\pm}$ and $\sigma_{(j)}^z$ (for $1\leqslant j\leqslant N$). They satisfy the commutation relations

$$\begin{aligned}&[\sigma_{(j)}^{\pm},\sigma_{(k)}^z]=\mp\sigma_{(j)}^{\pm}\delta_{jk}\\&[\sigma_{(j)}^+,\sigma_{(k)}^-]=2\sigma_{(j)}^z\delta_{jk}.\end{aligned} \tag{2}$$

If β is the amplitude of the driving field, then the scaled field α is given by

$$\alpha=-\mathrm{i}\left(\frac{2\varepsilon V}{\hbar\omega}\right)^{1/2}\mathrm{e}^{\mathrm{i}\phi_T}\frac{\sqrt{T}F}{\pi}\beta \tag{3}$$

where ϕ_T is the phase change at the input mirror on transmission, T is the transmission of the mirror at the output, F is the finesse of the cavity, κ is the cavity field decay rate, $\gamma_\parallel^{-1}$ is the radiative lifetime and γ_0 is the atomic decay rate due to collisions. $J^{\pm}$ and J^z are collective atomic operators

$$\begin{aligned}J^{\pm}&=\sum_{j=1}^{N}\mathrm{e}^{\pm\mathrm{i}k\cdot x}\sigma_{(j)}^{\pm}\\J^z&=\sum_{j=1}^{N}\sigma_{(j)}^z.\end{aligned} \tag{4}$$

If we now specialise to a bad cavity [11], we have

$$\gamma_\parallel,(\tfrac{1}{2}\gamma_\parallel+\gamma_0)\ll gN^{1/2}\qquad gN^{1/2}/\log N\ll\kappa \tag{5}$$

and it is possible to adiabatically eliminate the field and obtain an equation for the density $\rho_A(t)$ for the atomic system alone. $\rho_A(t)$ satisfies

$$\dot{\rho}_A(t)=-\mathrm{i}\hat{\Omega}[J^++J^-,\rho_A]+(g^2/\kappa)(2J^-\rho_AJ^+-J^+J^-\rho_A-\rho_AJ^+J^-)$$

$$+\tfrac{1}{2}\gamma_\parallel\sum_{j=1}^{N}(2\sigma_{(j)}^-\rho_A\sigma_{(j)}^+-\sigma_{(j)}^+\sigma_{(j)}^-\rho_A-\rho_A\sigma_{(j)}^+\sigma_{(j)}^-)$$

$$+\tfrac{1}{2}\gamma_0\sum_{j=1}^{N}(2\sigma_{(j)}^z\rho_A\sigma_{(j)}^z-\tfrac{1}{2}\rho_A) \tag{6}$$

and $\hat{\Omega} = -ig\alpha/\kappa$.

Before we proceed to the solution of the ME we will briefly outline the FPE approach [12, 13] and the decorrelation method [5, 14], so that we can draw inferences later concerning their validity. There is more than one FPE which can be associated with an ME, although they are all obtained via a truncation of a Kramers-Moyal type of expansion [7]. Inherent in an FPE is the ordering of operators in correlation functions which can be calculated using it. We will be interested in normal and symmetric orderings. The Fokker-Planck probability distribution P associated with the density matrix ρ of equation (1) can be defined by

$$\chi_\rho(\boldsymbol{\lambda}) = \int \prod \mathrm{d}^2\alpha_j \, P(\alpha_1, \ldots, \alpha_j, \ldots) \exp\left(\mathrm{i} \sum_j \lambda_j \alpha_j\right) \tag{7}$$

where

$$\chi_\rho = \mathrm{Tr}(\rho\Lambda)$$

and

$$\Lambda = \mathrm{e}^{\mathrm{i}\lambda_1 J^+} \mathrm{e}^{\mathrm{i}\lambda_2 J^z} \mathrm{e}^{\mathrm{i}\lambda_3 J^-} \mathrm{e}^{\mathrm{i}\lambda_4 a^+} \mathrm{e}^{\mathrm{i}\lambda_5 a}$$

for normal ordering whereas for symmetric ordering

$$\Lambda = \exp(\mathrm{i}(\lambda_1 J^+ + \lambda_2 J^z + \lambda_3 J^- + \lambda_4 a^+ + \lambda_5 a)).$$

The multi-variable FPE for P, which result from a truncation keeping derivatives to second order, are difficult to solve in general. A numerical solution is facilitated when the diffusion matrix is positive definite by noting the equivalence with Ito stochastic differential equations [7].

If $\boldsymbol{A}$ is the drift vector and $\mathbf{D}$ the diffusion matrix for the FPE then the associated Ito stochastic differential equation is

$$\mathrm{d}\boldsymbol{\alpha} = \boldsymbol{A}\,\mathrm{d}t + \mathbf{B}\,\mathrm{d}\boldsymbol{W} \tag{8}$$

where $\mathrm{d}W_1, \mathrm{d}W_2, \ldots, \mathrm{d}W_5$ are independent Wiener processes and $\mathbf{B}\mathbf{B}^{\mathrm{T}} = \mathbf{D}$. For P in the normally ordered and symmetric cases we will use the positive P [6, 12] and Wigner [13] representations, respectively. All 'physically' real quantities are allowed to be complex in the positive P representation and the stochastic trajectory solutions of equation (8) are free to explore all of a ten-dimensional phase space. It is to be hoped that in calculations the weight contributed by parts of phase space which very badly violate the 'physical' requirements on the inversion (namely $|z_5| \leqslant N/2$) should be small. We can proceed without rejecting any trajectories. At the critical point, however, $g^{(2)}(0)$ is estimated poorly since some instability in the positive P method exists which has recently been found [6]. This instability causes the trajectories to linger too long in parts of phase space which, in a conventional representation, would be unphysical. Lower-order moments, such as the expectation of the field and intensity, are not affected significantly by the instability. We have also adopted a more stringent criterion in which trajectories leaving $|z_5| \leqslant N/2$ are rejected and so, by construction, the instability is removed. For the Wigner case this instability is not present. However, $\mathbf{D}$ is not positive definite throughout phase space although it is so on the manifold of steady states. In our calculations at the critical point, the stochastic trajectories very rarely visited regions of phase space lacking this positive definiteness. At the critical point we have concentrated on the qualitative differences between the ME and FPE calculations. For some quantities the FPE results do not show the same qualitative trends as those for the ME. The Ito stochastic differential equations are solved numerically by essentially changing the infinitesimals to differences and the resulting algorithm [7] is

the analogue of the Euler method for ordinary differential equations. The algorithm is widely used and gives satisfactory results provided the time step is short enough. Independent noise sources are simulated by successive calls to a Gaussian distribution random number generator (Numerical Algorithms Group (NAG) routine G05DDF). The routine updates the seed of the underlying multiplicative congruential generator in each call, and thus the independence of the noise sources is maintained. The statistical properties of this generator are good for up to 2^{40} calls as it has an 80 bit seed. In a typical run we use the order of 10^9 random numbers and there should not be any problems with spurious correlations.

We will also need the results of the linearised analysis of fluctuations [3, 4]. In the bad cavity limit

$$g^{(2)}(0)-1=\frac{4N_sC}{\kappa x^2}\gamma_\perp(W_-+W_+) \tag{9}$$

where

$$W_\pm=\mp\frac{1}{2\lambda_\pm}\frac{x^2}{1+x^2}\frac{1}{\lambda_+^2-\lambda_-^2}[\lambda_\pm^2(\nu^{-1}-1)-\nu^{-2}(\nu^{-1}-1-2x^2)]$$

$$\lambda_\pm=\frac{1}{2}\left[2+\frac{y}{x}\pm\left(2-\frac{y}{x}\right)^2-8x(2x-y)^{1/2}\right]$$

and x is the steady-state output field related to the input y by

$$y=x\left(1+\frac{2C}{1+x^2}\right).$$

Moreover, the ratio of incoherent intensities is

$$\frac{4C^2}{N}\frac{x^2}{1+x^2}\left\{\frac{y}{x}\left[\frac{y}{x}+x^2\left(2-\frac{y}{x}\right)\right]\right\}^{-1}.$$

The other common approach to the solution of the ME is through a decorrelation method. We shall consider the master equation of (6). On taking moments of operators using the master equation we obtain

$$(\mathrm{d}/\mathrm{d}t)\langle J^-\rangle=2\mathrm{i}\hat{\Omega}\langle J^zJ^-\rangle+(2g^2/\kappa)\langle J^zJ^-\rangle-\gamma_\perp\langle J^-\rangle \tag{10}$$

$$(\mathrm{d}/\mathrm{d}t)\langle J^z\rangle=\mathrm{i}\hat{\Omega}(\langle J^-\rangle-\langle J^+\rangle)-(2g^2/\kappa)\langle J^+J^-\rangle-\gamma_\parallel(\tfrac{1}{2}N+\langle J^z\rangle). \tag{11}$$

If we calculate the time evolution of $\langle J^+J^-\rangle$, higher-order correlations will appear in the equations and in that way an infinite hierarchy of equations is obtained. In order to break this hierarchy, the scaling of collective fluctuations [3, 5, 14, 15] with respect to N is used. It is assumed that

$$\langle J^zJ^-\rangle-\langle J^z\rangle\langle J^-\rangle=\mathrm{O}(1/N)$$

and (12)

$$\langle J^+J^-\rangle-\langle J^+\rangle\langle J^-\rangle=\mathrm{O}(1/N).$$

This decorrelation certainly leads to a closed set of equations for $\langle J^-\rangle$ and $\langle J^z\rangle$. We will examine the quantities $(\langle J^zJ^-\rangle-\langle J^z\rangle\langle J^-\rangle)$ and $(\langle J^+J^-\rangle-\langle J^+\rangle\langle J^-\rangle)$ as a function of N in different parameter regions. In the calculation of the fluorescent spectrum, we need the correlation function $\langle\sigma_{(j)}^+(t)\sigma_{(j)}^-(0)\rangle$ and through the quantum regression

theorem [14] it is necessary to evaluate $\langle\sigma_{(i)}^{\mu}\sigma_{(j)}^{\nu}\rangle$ where $\mu, \nu = +, -, z$. If $i = j$ then from the explicit form of the Pauli matrices it is possible to express the product in terms of single Pauli matrices. If $i \neq j$ then owing to the permutation symmetry of the atomic dynamics we have

$$\langle\sigma_{(i)}^{\mu}\sigma_{(j)}^{\nu}\rangle_{i\neq j} = \frac{1}{N(N-1)}\langle J^{\mu}J^{\nu}\rangle - \frac{1}{N-1}\langle\sigma_{(i)}^{\mu}\sigma_{(i)}^{\nu}\rangle. \tag{13}$$

In particular

$$\langle\sigma_{(i)}^{z}\sigma_{(j)}^{-}\rangle_{i\neq j} - \langle\sigma^{z}\rangle\langle\sigma^{-}\rangle = \frac{1}{N(N-1)}(\langle J^{z}J^{-}\rangle - \tfrac{1}{2}\langle J^{-}\rangle) - \frac{\langle J^{z}\rangle\langle J^{-}\rangle}{N^{2}} \tag{14}$$

and

$$\langle\sigma_{(i)}^{+}\sigma_{(j)}^{-}\rangle_{i\neq j} - \langle\sigma^{+}\rangle\langle\sigma^{-}\rangle = \frac{1}{N(N-1)}(\langle J^{+}J^{-}\rangle - \langle J^{z}\rangle) - \frac{1}{2(N-1)} - \frac{\langle J^{+}\rangle\langle J^{-}\rangle}{N^{2}}. \tag{15}$$

Hence the knowledge of the decorrelation in (26) is enough to calculate the single-atom decorrelation too.

2. Direct solution of master equations

We will first concentrate on the bad cavity master equation given in equation (6). If $\gamma_{\parallel} = \gamma_0 = 0$ then the steady-state solution has been found exactly [9], and this is relevant to cooperative resonance fluorescence. For $N = 1$ the ME can be solved in detail [8]. However, the general case which is of most interest for bistability is not amenable to analytic treatment as far as we are aware. The ME is an operator equation. It can be converted easily into a coupled set of linear ordinary differential equations for the matrix elements of the density matrix in a suitable basis for the atomic states. An adequate basis for the atomic states of N two-level atoms is the N-fold direct product of states of the single-atom states. The number of density matrix elements grows as 2^{2N} which is just the square of the number of basis states. In the absence of single-atom damping terms, the Liouvillian commutes with the square of the collective angular momentum and so angular momentum or Dicke states [16] can be introduced. Since states with different total angular momentum do not mix in this case, the growth of the number of density matrix elements is proportional to N^2 which is of course a considerable reduction in number over that resulting from the use of the naive atomic-state basis. In the presence of single-atom decays, the total angular momentum is not a good quantum number. We have to find a symmetry of equation (6) even in the presence of single-atom decays. These decay terms do not single out any particular atom for special treatment. Consequently, if a density matrix is initially permutation symmetric, its evolution will preserve that symmetry. Moreover, since the ME has a unique steady-state solution, such a solution will be permutation symmetric. The approach which takes into account the simplification of permutation symmetry still uses density matrix elements expressed in the simple product basis but works with only a small subset of them. If $|\boldsymbol{i}\rangle$ denotes the state

$$|\boldsymbol{i}\rangle = |i_1\rangle|i_2\rangle \ldots |i_N\rangle \tag{16}$$

(where $i_j = \pm 1$ for $1 \leqslant j \leqslant N$ and $|-1\rangle$ is the ground state of an atom while $|1\rangle$ is the excited state of the atom) then the density matrix can be written as

$$\rho = \sum_{i,j} |i\rangle\langle j| \rho_{ij}. \tag{17}$$

If S_N denotes the permutation group over N objects and s is an element in S_N, then there is a unitary operator U_s acting on the density matrix elements such that

$$\rho = U_s \rho U_s^{-1} = \sum_{i,j} |s(i)\rangle\langle s(j)| \rho_{ij} \tag{18}$$

where i is the N-tuple $(i_1, i_2, \ldots, i_N)$, $s(i)$ is the N-tuple $(i_{s(1)}, i_{s(2)}, \ldots, i_{s(N)})$ and the permutation s is given by

$$\begin{pmatrix} 1 & 2 & \ldots & N \\ s(1) & s(2) & \ldots & s(N) \end{pmatrix}. \tag{19}$$

In particular, equation (18) implies that

$$\rho_{ij} = \rho_{s^{-1}(i)s^{-1}(j)} \tag{20}$$

where s^{-1} is the inverse element of s in S_N, and equation (20) is valid for arbitrary s. Since the numbers i_j are ± 1, an N-tuple i consists of some number n of $+1$ and $(N-n)$ of -1. A permutation s can be chosen such that

$$i_{s(j)} = \begin{cases} 1 & \text{for } 1 \leqslant j \leqslant n \\ -1 & \text{for } j > n. \end{cases} \tag{21}$$

Such an s is not unique. If t_n is a permutation in S_n then it can be extended to a permutation $\bar{t}_n$ of S_N with the definition

$$\bar{t}_n(i) = (i_{t(1)}, i_{t(2)}, \ldots, i_{t(n)}, i_{n+1}, i_{n+2}, \ldots, i_N). \tag{22}$$

Similarly, if t_{N-n} is a permutation in $S_{(N-n)}$, it can be extended to a permutation $\hat{t}_{N-n}$ of S_N by

$$\hat{t}_{N-n}(i) = (i_1, i_2, \ldots, i_n, i_{t(n+1)}, i_{t(n+2)}, \ldots, i_{t(N)}). \tag{23}$$

Hence if s satisfies (21) then so does $(t_n \cdot s)$, $(s \cdot \hat{t}_{N-n})$ and $(\hat{t}_n \cdot s \cdot \hat{t}_{N-n})$. In this way it is always possible to find a permutation s such that

$$i_{s^{-1}(k)} = \begin{cases} 1 & \text{for } 1 \leqslant k \leqslant n \\ -1 & \text{for } n < k \end{cases}$$

$$j_{s^{-1}(k)} = \begin{cases} 1 & \text{for } 1 \leqslant k \leqslant m \\ -1 & \text{for } m < k \leqslant n \\ 1 & \text{for } n < k \leqslant (n+m') \\ -1 & \text{for } (n+m') < k \leqslant N. \end{cases} \tag{24}$$

For an s with these properties we will write

$$\rho_{s^{-1}(i)s^{-1}(j)} = \rho_{n;m,m'}. \tag{25}$$

From the ME we can deduce equations for $\rho_{n;mm'}$ if we note that for any S_N symmetric operator O

$$O_{s^{-1}(j)s^{-1}(i)} = O_{m+m';m,n-m} \tag{26}$$

(where the notation has the obvious interpretation).

Moreover we can show that

$$(J^{-}\rho J^{+})_{n;m,m'} = m'(n-m)\rho_{n+1;m+2,m'-1} + (N-n-m')(N-n-m'-1)\rho_{n+1;m,m'+1} + (N-n-m')(n-m+m'+1)\rho_{n+1;m+1,m'} \tag{27}$$

and

$$(J^{-}\rho)_{n;m,m'} = m'\rho_{n+1;m+1,m'-1} + (N-n-m')\rho_{n+1;m,m'}. \tag{28}$$

Such relations help to establish the evolution equations which are given by

$$\begin{aligned}(\mathrm{d}/\mathrm{d}t)\rho_{n;m,m'} = {} & -\mathrm{i}\hat{\Omega}[(n-m)\rho_{n-1;m,m'} + m\rho_{n-1;m-1,m'+1} + (N-n-m')\rho_{n+1;m,m'} \\ & + m'\rho_{n+1;m+1,m'-1} - m\rho_{n;m-1,m'} - m'\rho_{n;m,m'-1} \\ & - (n-m)\rho_{n;m+1,m'} - (N-n-m')\rho_{n;m,m'+1}] \\ & + \gamma[(N-n-m')\rho_{n+1;m+1,m'} - \tfrac{1}{2}(n+m+m')\rho_{n;m,m'}] \\ & + (g^2/\kappa)\{2[(N-n-m')\rho_{n+1;m+1,m'} + m'(n-m)\rho_{n+1;m+2,m'-1} \\ & + [m'(N-n-m') + (N-n-m')(n-m)]\rho_{n+1;m+1,m'} \\ & + (N-n-m')(N-n-m'-1)\rho_{n+1;m,m'+1}] \\ & - [m'm + n + m + m' + (N-n-m')(n-m+m') + m(n-m)]\rho_{n;m,m'} \\ & - 2m(N-n-m')\rho_{n;m-1,m'+1} - 2m'(n-m)\rho_{n;m+1,m'-1}\}.\end{aligned} \tag{29}$$

Before we proceed further it is necessary to determine the number of separate $\rho_{n;m,m'}$ that we are considering. It is easy to see that this is

$$\sum_{n=0}^{N}(n+1)(N-n+1) = \tfrac{1}{6}(N+1)(N+2)(N+3). \tag{30}$$

Hence, although the number of density matrix elements for large N is a factor of the order of N larger than the number of elements required when we use the Dicke basis, our method can be used in the general situation whereas the Dicke basis cannot. With computers available to us, N up to 45 can be treated. This limit is imposed by the memory requirements of the Adam's method differential equation solver that we used. For 45 atoms we have 17 296 coupled differential equations and this requires 820 000 words of memory. The run times for the calculations are of the order of 35 min CPU time.

The FPE are solved for two parameter regimes. In one regime ($C=2$, $y=\sqrt{2}$) the linearised analysis would hold (if N were large). In the other ($C=4$, $y=3\sqrt{3}$) linearised analysis does not exist, however large N is. Since the ME is solved for $N \leqslant 45$ and the FPE holds for large N, we need to solve the Ito stochastic differential equations (SDE) for as small an N as possible. In figure 1 we consider the expectation value of the intensity for $C=2$ and $y=2^{1/2}$. If N were large, then this situation would be characterised by no bistability and small intracavity fields. The points calculated from the ME lie on a smooth 'curve'. The FPE results were found from the equivalent SDE. We note that the asymptotic theories are good even to $N=45$. The calculation for $N=100$ shows that, in the positive-P SDE calculation, the rejection of trajectories which leave the physical region overestimates $\langle I \rangle$. For $N=200$ and 400 the results are insensitive to the rejection criterion for trajectories. It is not possible to calculate quantities using the Wigner SDE since the problem of non-positive definiteness of the

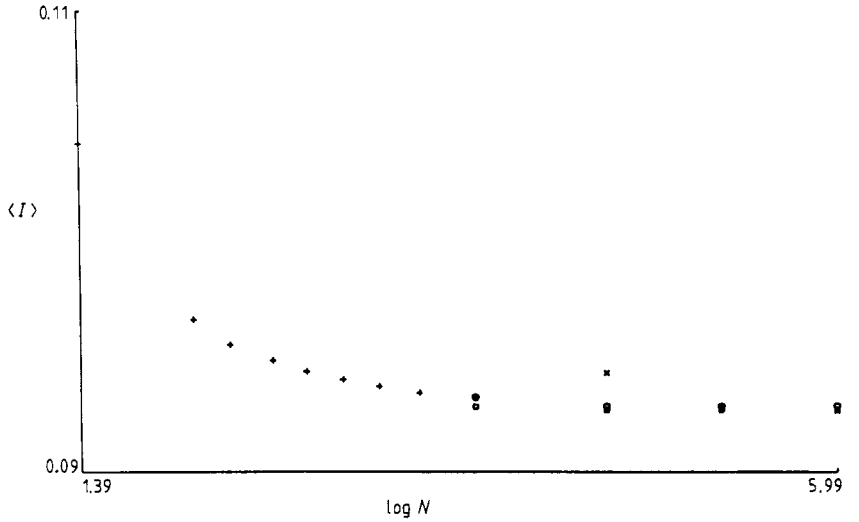

Figure 1. $\langle I \rangle$ against N, where I is $\langle \hat{x}^* \hat{x} \rangle$, for $C = 2$ and $y = \sqrt{2}$. +, ME; □, linearised analysis; ×, FPE positive P with rejection criterion; ○, FPE positive P.

diffusion matrix is acute. There is agreement to 0.4% even for $N = 32$ between ME and the linearised analyses. We have also found in other cases that, where a linearised FPE analysis is valid for large N, then the linearised analysis when extrapolated down to small N does as well and often better than the non-linear treatment. There is, of course, no reason to have supposed that the FPE would have performed better than the linearised analysis since both are asymptotic theories. It is interesting to see that the more simple version of the asymptotic theory extrapolates better down to small N.

In figure 2 we consider again $C = 2$ and $Y = 2^{1/2}$, but this time $N(1 - g^{(2)}(0))$ is calculated. The factor of N is included since linearised theory implies that to leading

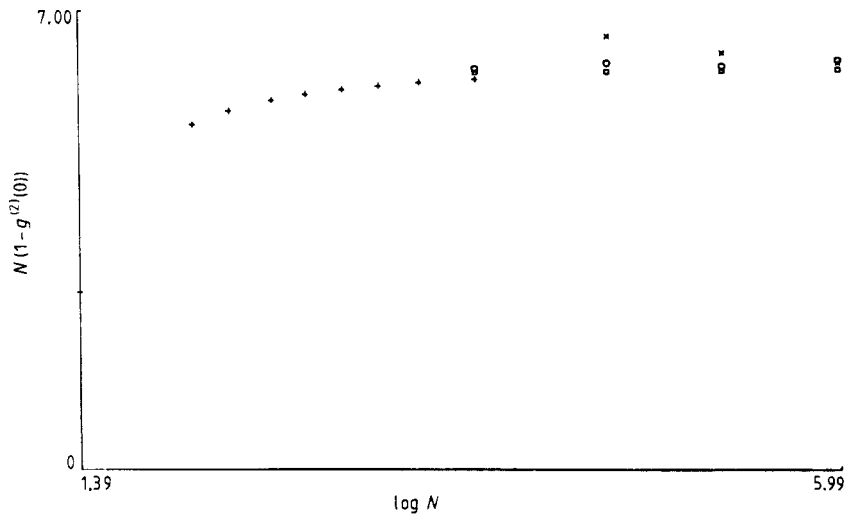

Figure 2. $N(1 - g^{(2)}(0))$ against N for $C = 2$ and $y = \sqrt{2}$ (symbols as in figure 1).

order $(1-g^{(2)}(0))$ is of order $1/N$. The linearised theory and the SDE (when no trajectories are rejected) show good agreement with the ME again even for N as low as 45. Hence even for quite small numbers of atoms the asymptotic theories are quantitatively reliable for non-critical points.

We will now turn to the regime where linearised analysis is invalid for all values of N. From the formulae of linearised analysis we note that $C=4$ with $y=3\sqrt{3}$ is such a case. In the semiclassical picture, hysteresis is about to start at this parameter value. An infinitesimal increase in C would lead to an infinitesimal hysteresis loop. We first examine $g^{(2)}(0)$ (see figure 3). Again we find good agreement between the ME and FPE calculations. However, the positive-P calculations with the procedure of no rejection of trajectories used earlier give too high values for $g^{(2)}(0)$. As we have already mentioned, this is due to an instability of the positive-P equations at the critical point. Lower-order moments of the field are not sensitive to criteria for dealing with trajectories which assume large values. Owing to the instability in the positive-P FPE for the critical point, it is necessary to have an independent FPE calculation. The Wigner FPE shows good agreement with the ME result for $N=45$, and for the larger N values gives a smooth extrapolation of the ME results. We conclude that the FPE and ME are again in good agreement even up to $N=45$. However, this is not so for $|\langle\hat{x}\rangle|$ and $\langle I\rangle$. From figure 4 we discover the unexpected feature that the modulus of the expectation of the intracavity field x decreases below the deterministic value of $3^{1/2}$. This drop in value increases as N becomes smaller. Both the ME and FPE show a drop although the FPE results in no sense join smoothly onto the ME ones. The Wigner FPE gives the results closest to those of the ME, but for $N=45$ there is a 5% difference while for the positive-P calculation the difference is as much as 12–13%. When we examine figure 5, the FPE and ME have different trends. For the ME $\langle I\rangle$ decreases smoothly for increasing N and tends towards the deterministic value of 3. The FPE calculations of the same quantity show it increasing towards 3 from below. This does not mean that we are in a region of non-classical statistics since $(\langle\hat{x}^*\hat{x}\rangle-\langle\hat{x}\rangle\langle\hat{x}^*\rangle)$ is positive. For $N=45$ the Wigner FPE and ME agree to 8%, whereas the disagreement between the

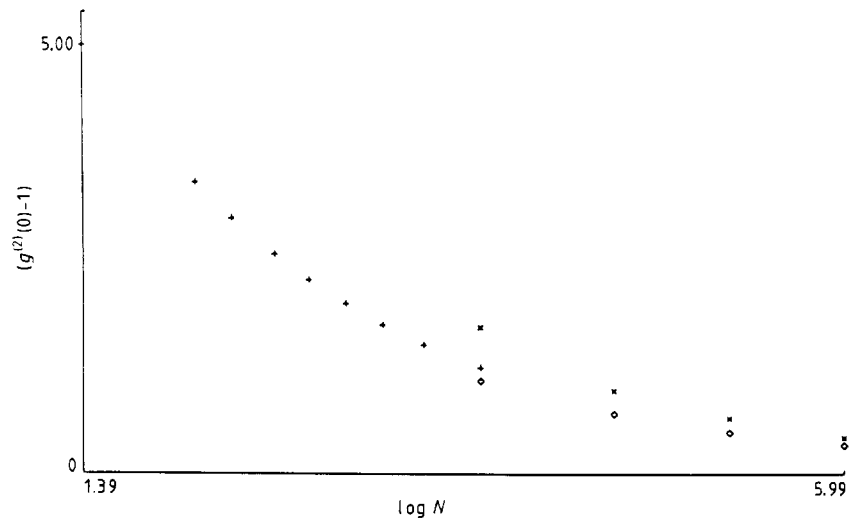

Figure 3. $(g^{(2)}(0)-1)$ against N for $C=4$ and $y=3\sqrt{3}$. +, ME; ◇, FPE Wigner; ×, FPE positive P with rejection criterion.

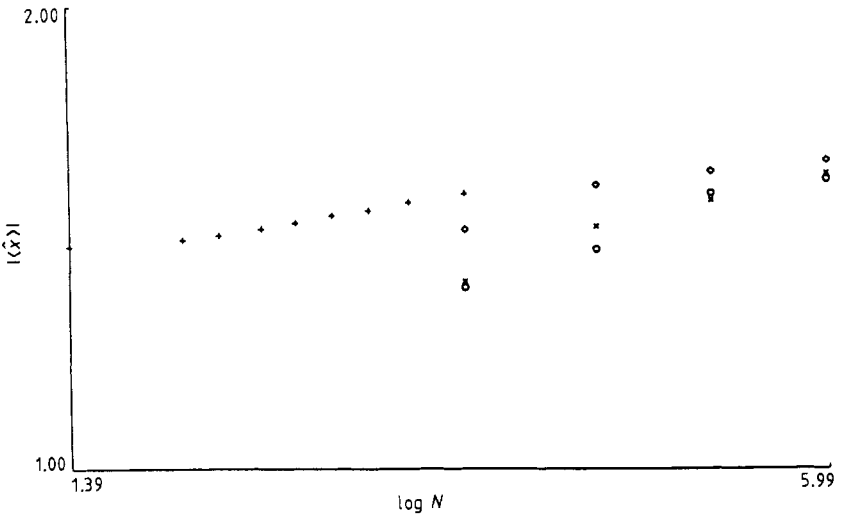

Figure 4. $|\langle\hat{x}\rangle|$ against N for $C=4$ and $y=3\sqrt{3}$ (symbols as in figures 1 and 3).

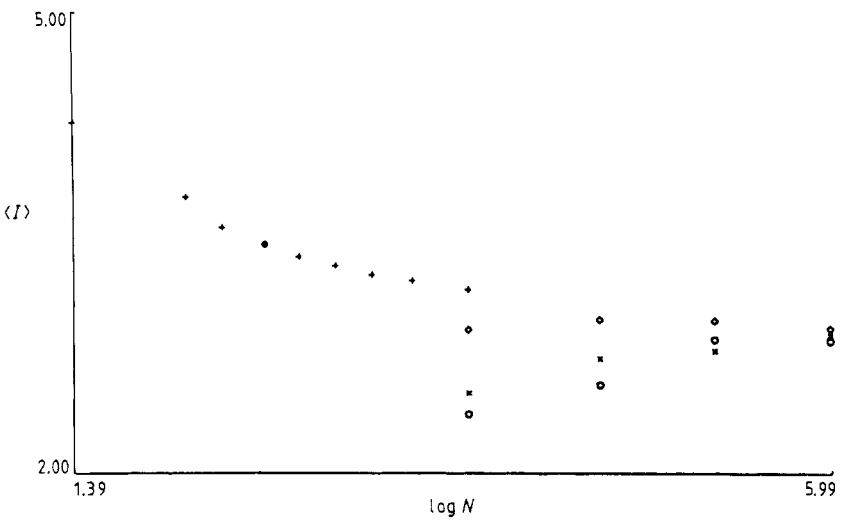

Figure 5. $\langle I\rangle$ against N for $C=4$ and $y=3\sqrt{3}$ (symbols as in figures 1 and 3).

positive-P FPE and the ME is worse than 20%. In addition there is the qualitative difference of the approach to the deterministic value, i.e. the ME from above and the FPE from below. Hence we can state that the FPE are qualitatively incorrect in this particular case.

We expect decorrelation assumptions to be maximally violated for $C=4$ and $y=3\sqrt{3}$, and so we have evaluated quantities for this case for $N=4$-45. Our aim has been to find scaling behaviour in N. Linearised analysis (inapplicable here) typically requires $(\langle J^zJ^-\rangle-\langle J^z\rangle\langle J^-\rangle)$ to be proportional to N. We find a proportionality of $N^{1.55}$ approximately. Similarly, $(\langle J^+J^-\rangle-\langle J^+\rangle\langle J^-\rangle)$ behaves like $N^{1.53}$ roughly. Scaling is not exact. If we consider the normalised (intensive) quantities $\hat{J}^z$ $(=J^z/N)$ and $\hat{J}^{\pm}$

$(=J^{\pm}/N)$ then we can state that approximately

$$\langle \hat{J}^z \hat{J}^- \rangle - \langle \hat{J}^z \rangle \langle \hat{J}^- \rangle \sim N^{-1/2}$$
$$\langle \hat{J}^+ \hat{J}^- \rangle - \langle \hat{J}^+ \rangle \langle \hat{J}^- \rangle \sim N^{-1/2}. \tag{31}$$

Moreover, we find that $(\langle \hat{x}^* \hat{x} \rangle - \langle \hat{x} \rangle \langle \hat{x}^* \rangle)$ has a $N^{-\alpha}$ dependence where α varies between $[0.425, 0.5]$ as N increases, which is similar to (31). The scaling of fluctuations as $N^{-1/2}$ at the critical point has been found earlier [5] from the FPE, and can be derived analytically for the special case of one-dimensional FPE.

In conclusion we see that the extrapolation of the commonly used 'large N' theories for small N is surprisingly good in some situations. However, even for the same parameter values, different correlations are not uniformly well calculated by the FPE. There are also situations where the fluctuations are incorrectly treated at the qualitative level. The picture is not completely clear cut but those who use the large-N theories may expect to get some useful results even for the smallest systems for the currently planned experiments.

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